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Existence of the Thermodynamic Limit in Nonequilibrium Systems

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The existence of a thermodynamic limit in nonequilibrium stochastic and quantal systems is proven for finite-range interactions and macrovariables which are bounded in the sense of norm. This condition is easily confirmed to be satisfied for specific models, such as the kinetic Ising model and quantal spin systems.

KEY WORDS: Thermodynamic limit; nonequilibrium systems; stochastic process; kinetic Ising model; macrovariable; extensivity; quantum system; Banach algebra.

1. INTRODUCTION

One of the general concepts in nonequilibrium systems is the extensivity of a macrovariable and of the logarithm of a reduced density matrix or generating function, which was proposed by Kubo[1] for a Markovian process described by the Kramers–Moyal equation. This extensive property is a direct consequence of the existence of a thermodynamic limit in nonequilibrium systems, as discussed in previous papers.[2–5] The previous proof of the existence of a thermodynamic limit was given under the "abstract" condition that all relevant operators such as the Hamiltonian and a macrovariable **X** are local (or finite-range) operators bounded in the sense of certain canonical averages.[2–5] At first sight, it seems difficult to confirm this abstract condition in specific models.

The purpose of this paper is to give a rigorous proof of the existence of a thermodynamic limit in nonequilibrium systems, under the general *explicit* condition that all relevant operators are of finite range and bounded in the sense of norm (i.e., operators in a Banach algebra). The method of the proof

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is essentially the same as before.[2–5] An important new point in the present paper is to transform upper bounds appearing in the proof into some symmetric forms which are convenient to evaluate explicitly in terms of norms of relevant operators, as already reported briefly.[6,7]

In Section 2, a general scheme of the existence proof is given. Explicit evaluations of upper bounds in the proof are performed in Section 3 for stochastic systems and in Section 4 in quantal systems.

2. EXISTENCE THEOREM AND GENERAL SCHEME OF THE PROOF

A general nonequilibrium situation is described (i) by the following microscopic master equation for the probability function $P(t)$

$$\frac{\partial}{\partial t} P(t) = \Gamma(t)P(t) \tag{1}$$

for stochastic systems, where $\Gamma(t)$ is a temporal evolution operator at time t, and (ii) by the following Liouville equation for the density matrix

$$\frac{\partial}{\partial t} \rho(t) = \frac{1}{i}[\mathscr{H}(t), \rho(t)] \equiv \frac{1}{i}\, \delta_{\mathscr{H}(t)}\rho(t) \tag{2}$$

for quantal systems, where $\hbar = 1$ and $\delta_{\mathscr{H}(t)}$ is a so-called inner derivation in mathematics [cf. $\delta_{\mathscr{H}(t)} \equiv \mathscr{H}^{\times}(t)$ (Kubo's notation) in statistical mechanics[8]].

As in the previous papers,[2–5] it is convenient to introduce a generating function $\Psi(\lambda, t)$ defined by

$$\Psi_{\Omega}(\lambda, t) = \begin{cases} \mathrm{Tr}\, e^{\lambda X}\rho(t) & \text{(quantal)} \\ \sum e^{\lambda X}\rho(t) & \text{(stochastic)} \end{cases} \tag{3}$$

for volume Ω. The probability distribution function $\rho(X, t)$ of a macrovariable $\mathbf{X}$ is given by the inverse transformation

$$\rho(X, t) \equiv \mathrm{Tr}\, \delta(\mathbf{X} - X)\rho(t) = \frac{1}{2\pi i}\int_{c - i\infty}^{c + i\infty} e^{-\lambda X}\Psi_{\Omega}(\lambda, t)\, d\lambda \tag{4}$$

The extensivity property of $\rho(X, t)$ is that

$$\rho(X, t) \simeq C \exp[\Omega\phi(x, t)] \tag{5}$$

for large volume Ω. This extensive property can be derived[2–5] from the extensivity of the generating function; i.e.,

$$\Psi_{\Omega}(\lambda, t) \simeq C' \exp[\Omega\psi(\lambda, t)] \tag{6}$$

for large volume Ω, namely the fact that

$$\lim_{\Omega \to \infty} \frac{1}{\Omega} \log \Psi_\Omega(\lambda, t) \equiv \psi(\lambda, t) \tag{7}$$

exists. Here, we prove the existence of the thermodynamic limit of the generating function. However, it is more convenient to introduce the following generating function:

$$\Psi_\Omega(\lambda, \mu, t) \equiv \mathrm{Tr}(\exp \lambda \mathbf{X}) V(t)[\exp(\tfrac{1}{2}\mu \mathbf{Y}) \exp(-\beta^{(i)}\mathscr{H}^{(i)}) \exp(\tfrac{1}{2}\mu \mathbf{Y})] \tag{8}$$

for two macrovariables $\mathbf{X}$ and $\mathbf{Y}$, where $\rho(0) = \exp(-\beta^{(i)}\mathscr{H}^{(i)})$ and

$$\begin{aligned} V(t) &= \exp_+ \int_0^t \mathscr{L}(s)\, ds \\ &= 1 + \sum_{n=1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \mathscr{L}(t_1) \cdots \mathscr{L}(t_n) \end{aligned} \tag{9}$$

with

$$\mathscr{L}(t) = \begin{cases} \Gamma(t) & \text{(stochastic)} \\ -i\delta_{\mathscr{H}(t)} & \text{(quantal)} \end{cases} \tag{10}$$

For the details of the ordered exponential $\exp_+(\cdots)$, see the paper by Kubo.[9] The time correlation function $\langle \mathbf{X}(t)\mathbf{Y}(0)\rangle_t$ in equilibrium and nonequilibrium systems can be obtained by differentiating the above generating function $\Psi(\lambda, \mu, t)$ with respect to λ and μ. In this sense, the existence proof of $\Psi(\lambda, \mu, t)$ in this paper may also give a mathematical foundation to Kubo's linear response theory[8,10] in the thermodynamic limit. Clearly we have $\Psi_\Omega(\lambda, 0, t) = \Psi_\Omega(\lambda, t)$.

In order to prove the existence of the thermodynamic limit

$$\lim_{\Omega \to \infty} \frac{1}{\Omega} \log \Psi_\Omega(\lambda, \mu, t) \tag{11}$$

we consider systems of increasing size L_n (say, $L_n = 2^n a$, where n is a large integer and $\Omega_n = L_n{}^d$), as in static arguments[11-13] and as in Refs. 2–4. Correspondingly, we define $\psi_n(\lambda, \mu, t)$ by

$$\psi_n(\lambda, \mu, t) = \Omega_n^{-1} \log \Psi_{\Omega_n}(\lambda, \mu, t) \tag{12}$$

As in Refs. 2–4, our main task is to prove that this series of functions $\{\psi_n(\lambda, \mu, t)\}$ satisfies Cauchy's condition on convergence. For this purpose, we divide the volume Ω_n into 2^d subdomains Ω_{n-1} and provide each domain with an inside margin of width b (the range of local operators) as shown in Fig. 1. Each margined domain of Ω_{n-1} is denoted by $\hat{\Omega}_{n-1}$ (i.e., the volume

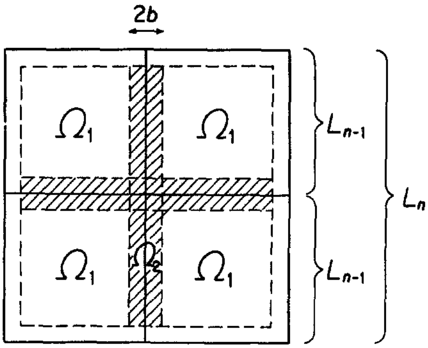

Fig. 1. Domains Ω_1 with inside margins of width b; Ω_2 is the shaded region.

$\hat{\Omega}_n = \hat{L}_n{}^d$; $\hat{L}_n = L_n - 2b$). Thus, we redefine $\Psi_{\Omega_n}(\lambda, \mu, t)$ by (8) with $\mathbf{X}$, $\mathbf{Y}$, $\mathscr{H}^{(i)}$, and $\mathscr{H}(t)$ [or $\Gamma(t)$] defined by the integrals

$$\mathbf{X} = \int \mathbf{X}(\mathbf{r})\, d\mathbf{r}, \qquad \mathbf{Y} = \int \mathbf{Y}(\mathbf{r})\, d\mathbf{r}, \qquad \mathscr{H}^{(i)} = \int \mathscr{H}^{(i)}(\mathbf{r})\, d\mathbf{r}$$

and

$$\mathscr{H}(t) = \int \mathscr{H}(\mathbf{r}, t)\, d\mathbf{r} \qquad \text{or} \qquad \Gamma(t) = \int \Gamma(\mathbf{r}, t)\, d\mathbf{r} \tag{13}$$

over the domain $\hat{\Omega}_n$. Let us call the boundary region shaded in Fig. 1 domain Ω_2 and the rest we call domain Ω_1, as before. That is, $\hat{\Omega}_n = \Omega_1 + \Omega_2$. Then, we separate each of the operators $\mathbf{X}$, $\mathbf{Y}$, $\mathscr{H}^{(i)}$, $\mathscr{H}(t)$, and $\Gamma(t)$ into two parts:

$$\begin{aligned} &\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2, \qquad \mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2, \qquad \mathscr{H}^{(i)} = \mathscr{H}_1^{(i)} + \mathscr{H}_2^{(i)} \\ &\mathscr{H}(t) = \mathscr{H}_1(t) + \mathscr{H}_2(t) \qquad \text{and} \qquad \Gamma(t) = \Gamma_1(t) + \Gamma_2(t) \end{aligned} \tag{14}$$

where

$$\mathbf{X}_j = \int_{\Omega_j} \mathbf{X}(\mathbf{r})\, d\mathbf{r}, \qquad \mathscr{H}_j^{(i)} = \int_{\Omega_j} \mathscr{H}^{(i)}(\mathbf{r})\, d\mathbf{r}, \qquad \mathscr{H}_j(t) = \int_{\Omega_j} \mathscr{H}(\mathbf{r}, t)\, d\mathbf{r}, \quad \text{etc.} \tag{15}$$

Now, the key point for the proof of the existence of the thermodynamic limit is to evaluate the difference Δ_n between the two generating functions corresponding to $\Omega_1 + \Omega_2$ and Ω_1:

$$\begin{aligned} \Delta_n &\equiv \log \Psi_{\Omega_1 + \Omega_2}(\lambda, \mu, t) - \log \Psi_{\Omega_1}(\lambda, \mu, t) \\ &= \int_0^1 d\xi \frac{d}{d\xi} \log \Psi_\xi = \int_0^1 d\xi\, \Psi_\xi^{-1} \frac{d}{d\xi} \Psi_\xi \end{aligned} \tag{16}$$

where Ψ_ξ is defined by

$$\begin{aligned}\Psi_\xi &= \Psi_{\Omega_1+\xi\Omega_2}(\lambda, \mu, t)\\ &= \mathrm{Tr}[\exp(\lambda \mathbf{X}_\xi)]\left[\exp_+ \int_0^t \mathscr{L}_\xi(s)\, ds\right] \exp(\tfrac{1}{2}\mu \mathbf{Y}_\xi) \exp \mathscr{H}_\xi^{(i)} \exp(\tfrac{1}{2}\mu \mathbf{Y}_\xi)\end{aligned} \tag{17}$$

with

$$\mathbf{X}_\xi = \mathbf{X}_1 + \xi\mathbf{X}_2, \qquad \mathscr{H}_\xi^{(i)} = \mathscr{H}_1^{(i)} + \xi\mathscr{H}_2^{(i)}, \qquad \mathscr{L}_\xi(t) = \mathscr{L}_1(t) + \xi\mathscr{L}_2(t) \tag{18}$$

In the succeeding sections, we derive the following inequalities:

$$|\Delta_n| \leqslant |\lambda|\, \|\mathbf{X}_2\| + |\mu|\, \|\mathbf{X}_2\| + \|\mathscr{H}_2\| + \int_0^t \|\Gamma_2(s)\|\, ds \tag{19}$$

for stochastic systems and

$$|\Delta_n| \leqslant \Omega_2 \times \text{const} \tag{20}$$

for quantal systems. Here the norm of $\mathbf{X}$, $\mathbf{Y}$, $\mathscr{H}^{(i)}$, and $\mathscr{H}(t)$ is defined by the maximum value of the absolute magnitude of a stochastic variable for stochastic systems and by that of eigenvalues of an operator for quantal systems, and the norm of $\Gamma(t)$ is defined by

$$\|\Gamma(t)\| = \max_{P>0} |\Gamma(t)P|/P \tag{21}$$

The derivation of inequalities (19) and (20) is the main task of the present paper. These inequalities yield

$$|\psi_n(\lambda, \mu, t) - \psi_{n-1}(\lambda, \mu, t)| \leqslant 2^{-n}c(\lambda, \mu, t) \tag{22}$$

with the use of the facts that $\Omega_2 = (2bd)\hat{L}_n^{d-1}$ + (higher) and $\hat{\Omega}_n = \hat{L}_n{}^d = 2^{nd}a^d$ + (higher), where

$$c(\lambda, \mu, t) = \frac{2bd}{a}\left\{|\lambda|\, \|\mathbf{X}(\mathbf{r})\| + |\mu|\, \|\mathbf{Y}(\mathbf{r})\| + \|\mathscr{H}(\mathbf{r})\| + \int_0^t \|\Gamma(\mathbf{r}, s)\|\, ds\right\} \tag{23}$$

for a stochastic system and it is rather complicated for quantal systems. The physical meaning of inequality (22) is that the boundary effect can be neglected compared with the bulk part for large volume. Repeated application of (22) yields

$$|\psi_{n+m}(\lambda, \mu, t) - \psi_n(\lambda, \mu, t)| \leqslant 2^{-n}c(\lambda, \mu, t) \tag{24}$$

for any positive integer m. This is nothing but Cauchy's condition of the uniform convergence of the series $\psi_n(\lambda, \mu, t)$ for $|\lambda| \leqslant \Lambda$ (fixed), $|\mu| \leqslant M$

(fixed), and finite t, if $\mathbf{X}(\mathbf{r})$, $\mathbf{Y}(\mathbf{r})$, $\mathscr{H}^{(i)}(\mathbf{r})$, and $\mathscr{H}(\mathbf{r}, t)$ [or $\Gamma(\mathbf{r}, t)$] are operators in a Banach algebra (i.e., bounded in the sense of norm). Thus, this series possesses a well-defined limit $\psi(\lambda, \mu, t)$ as $n \to \infty$ (i.e., $\Omega \to \infty$) for t fixed, and furthermore we have

$$|\psi(\lambda, \mu, t) - \psi_n(\lambda, \mu, t)| \leqslant 2^{-n} c(\lambda, \mu, t) \tag{25}$$

The limit $\psi(\lambda, \mu, t)$ obtained for the above particular sequence of squares is also obtained for an arbitrary sequence of squares with edge increasing to infinity as in the static proof[11–13] of the thermodynamic limit of free energy. Furthermore, the above limiting process is easily extended to that of van Hove's sense, as in static cases.[11–13] Hence the following theorems hold.

Theorem 1 (stochastic). If Hermitian operators (or scalar) $\mathbf{X}$, $\mathbf{Y}$, $\mathscr{H}^{(i)}$, and temporal evolution operator $\Gamma(t)$ are translationally invariant sets of local operators in a Banach algebra, then the thermodynamic limit

$$\lim_{\Omega \to \infty} \frac{1}{\Omega} \log \Psi_\Omega(\lambda, \mu, t) \tag{26}$$

exists in the sense of van Hove, for $|\lambda| \leqslant \Lambda$, $|\mu| \leqslant M$, and finite t.

Theorem 2 (quantal). The limit (26) exists with the same conditions as Theorem 1 with the additional conditions that $|\lambda| < \lambda_0$, $|\mu| < \mu_0$, $t < t_0$, and $\beta^{(i)} < \beta_0^{(i)}$ for appropriate positive constants λ_0, μ_0, t_0, and $\beta_0^{(i)}$.

3. PROOF IN STOCHASTIC SYSTEMS

As the variables $\mathbf{X}$, $\mathbf{Y}$, and $\mathscr{H}^{(i)}$ are all scalar in stochastic systems, the generating function Ψ_ξ can be written as

$$\Psi_\xi = \sum_{\text{config}} \exp(\lambda \mathbf{X}_\xi) V_\xi(t) \exp(-\beta^{(i)} \mathscr{H}_\xi^{(i)} + \mu \mathbf{Y}_\xi) \tag{27}$$

where

$$V_\xi(t) = \exp_+ \int_0^t \{\Gamma_1(s) + \xi \Gamma_2(s)\}\, ds \tag{28}$$

This $V_\xi(t)$ is convergent for $\|\Gamma\| < \infty$.

Now, the quantity Δ_n in (15) is expressed as

$$\begin{aligned}\Delta_n = \int_0^1 d\xi\, \Psi_\xi^{-1} \sum_{\text{config}} [&\lambda \mathbf{X}_2 (\exp \lambda \mathbf{X}_\xi) V_\xi(t) \exp(-\beta^{(i)} \mathscr{H}_\xi^{(i)} + \mu \mathbf{Y}_\xi) \\ &+ (\exp \lambda \mathbf{X}_\xi) V_\xi(t) \{(-\beta^{(i)} \mathscr{H}_2^{(i)} + \mu \mathbf{Y}_2) \exp(-\beta^{(i)} \mathscr{H}_\xi^{(i)} + \mu \mathbf{Y}_\xi)\} \\ &+ (\exp \lambda \mathbf{X}_\xi) V_\xi(t) \int_0^t V_\xi^\dagger(s) \Gamma_2(s) V_\xi(s)\, ds \exp(-\beta^{(i)} \mathscr{H}_\xi^{(i)} + \mu \mathbf{Y}_\xi)]\end{aligned} \tag{29}$$

where we have used the following lemma.

Lemma 1. If $\mathscr{L}(t)$ depends upon the parameter ξ in (9), then we have

$$\frac{d}{d\xi} V(t) = V(t) \int_0^t V^\dagger(s) \frac{\partial \mathscr{L}(s)}{\partial \xi} V(s)\, ds$$
$$= \int_0^t V(s) \frac{\partial \mathscr{L}(s)}{\partial \xi} V^\dagger(s) V(t)\, ds \tag{30}$$

The proof of this lemma is given by multiplying $V^{-1}(t)$ on both sides from the left side (or right side for the second equation) and differentiating it with respect to t, with the use of the properties

$$V^\dagger(t) = V^{-1}(t) = \exp_-\left\{-\int_0^t \mathscr{L}(s)\, ds\right\}$$
$$= 1 + \sum_{n=1}^{\infty} (-1)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, \mathscr{L}(t_n) \cdots \mathscr{L}(t_1) \tag{31}$$

$$\frac{d}{dt} V(t) = \mathscr{L}(t) V(t) \tag{32}$$

$$\frac{d}{dt} V^\dagger = -V^\dagger(t) \mathscr{L}(t) \tag{33}$$

In order to evaluate explicitly the upper bound of $|\Delta_n|$, we note the following lemma.[2-5]

Lemma 2. If $f \leqslant g$ for any state, then $V_\xi(t) f \leqslant V_\xi(t) g$ for $t \geqslant 0$ and $0 \leqslant \xi \leqslant 1$.

This is easily seen from the fact that if $h \geqslant 0$, then $V_\xi(t) h \geqslant 0$ for $t \geqslant 0$.

By applying this lemma to (29), we obtain

$$|\Delta_n| \leqslant \int_0^1 d\xi \left\{ |\lambda|\, \|\mathbf{X}_2\| + \|-\beta^{(i)} \mathscr{H}_2^{(i)} + \mu \mathbf{Y}_2\| + \int_0^t \|V_\xi^\dagger(s) \Gamma_2(s) V_\xi(s)\|\, ds \right\}$$
$$\leqslant |\lambda|\, \|\mathbf{X}_2\| + |\mu|\, \|\mathbf{Y}_2\| + \beta^{(i)} \|\mathscr{H}_2^{(i)}\| + \int_0^t \|\Gamma_2(s)\|\, ds$$
$$\leqslant \Omega_2 \left\{ |\lambda|\, \|\mathbf{X}(\mathbf{r})\| + |\mu|\, \|\mathbf{Y}(\mathbf{r})\| + \beta^{(i)} \|\mathscr{H}^{(i)}(\mathbf{r})\| + \int_0^t \|\Gamma(\mathbf{r}, s)\|\, ds \right\} \tag{34}$$

Here we have made use of the following simple lemma.

Lemma 3. For any unitary operator U and Hermitian operator Q, we have

$$\|UQU^{-1}\| = \|Q\| \tag{35}$$

Thus, we arrive at the desired inequality (19). Hence Theorem 1 holds in stochastic systems. In particular, for a time-independent temporal evolution operator Γ, we have

$$\int_0^t \|\Gamma(\mathbf{r}, s)\| \, ds = t\|\Gamma(\mathbf{r})\| \tag{36}$$

Finally, we show that the above condition on the boundedness of the relevant operators is easily satisfied for specific models. For example, we consider the kinetic Ising model[14,15] whose temporal evolution operator is defined by

$$\Gamma(\mathbf{r}, t)f(\{\sigma_j\}) = -W_{\mathbf{r}}(\sigma_{\mathbf{r}}, t)f(\{\sigma_j\}) + W_{\mathbf{r}}(-\sigma_{\mathbf{r}}, t)f(..., -\sigma_{\mathbf{r}}, ...) \tag{37}$$

with a transition probability $W_{\mathbf{r}}(\sigma_{\mathbf{r}}, t)$. Clearly, we have that $\mathbf{X}(\mathbf{r})$, $\mathbf{Y}(\mathbf{r})$, and $\mathscr{H}^{(i)}(\mathbf{r})$ are bounded in ordinary situations; i.e., $\mathbf{X}(\mathbf{r}) = \sigma_{\mathbf{r}} = \pm 1$ (local magnetization), or $\mathbf{X}(\mathbf{r}) = J\sum_{\Delta} \sigma_{\mathbf{r}}\sigma_{\mathbf{r}+\Delta}$ (local energy, and $|\mathbf{X}(\mathbf{r})| \leqslant z|J|$, where z denotes the number of nearest neighbors) and also $\mathscr{H}^{(i)}(\mathbf{r}) = -\beta^{(i)} \times \sum_{\langle jk \rangle} J_{jk}^{(i)}\sigma_j\sigma_k$ (bounded), etc. The norm $\|\Gamma(\mathbf{r}, t)\|$ is shown to be bounded as

$$\|\Gamma(\mathbf{r}, t)\| = \max_{f>0} |\Gamma(\mathbf{r}, t)f|/f \leqslant \max_{\sigma_{\mathbf{r}}} W_{\mathbf{r}}(\sigma_{\mathbf{r}}, t) < \infty \tag{38}$$

The transition probability in (38) is finite from the definition of the model.

4. PROOF IN QUANTAL SYSTEMS

The relevant generating function in quantal systems in (17) is expressed as

$$\Psi_\xi = \mathrm{Tr}(\exp \lambda \mathbf{X}_\xi)U(t)[\exp(\tfrac{1}{2}\mu\mathbf{Y}_\xi) \exp(-\beta^{(i)}\mathscr{H}_\xi^{(i)}) \exp(\tfrac{1}{2}\mu\mathbf{Y}_\xi)]U^\dagger(t) \tag{39}$$

where

$$U_\xi(t) = \exp_+\left\{\frac{1}{i}\int_0^t \mathscr{H}_\xi(s)\, ds\right\}$$
$$= 1 + \sum_{n=1}^{\infty} \left(\frac{1}{i}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n\, \mathscr{H}_\xi(t_1) \cdots \mathscr{H}_\xi(t_n) \tag{40}$$

This $U_\xi(t)$ is convergent for $\|\mathscr{H}_\xi(t)\| < \infty$. Here, we have used the following well-known formula on the inner derivation $\delta_{\mathscr{H}}$:

$$e^{\delta_{\mathscr{H}}} Q = e^{\mathscr{H}} Q e^{-\mathscr{H}} \tag{41}$$

or more generally

$$\exp_+(\delta_{\mathscr{H}(t)})Q = \{\exp_+ \mathscr{H}(t)\}Q \exp_-\{-\mathscr{H}(t)\} \tag{42}$$

In order to differentiate Ψ_ξ with respect to ξ, it is convenient to note the following lemma.[2-4]

Lemma 4.

$$\frac{d}{dt} e^{A(x)} = \int_0^1 e^{(1-s)A(x)} A'(x) e^{sA(x)}\, dx = \int_0^1 e^{sA(x)} A'(x) e^{(1-s)A(x)}\, dx \tag{43}$$

This is a direct consequence of Lemma 1 by the transformation of variables $s = s't$ for a time-independent operator $\mathscr{L}$ [i.e., $A(x)$].

Using Lemmas 1 and 4, and rearranging the order of operators inside the trace operation, we arrive at the following convenient symmetric expression for Δ_n:

$$\begin{aligned}
\Delta_n = \int_0^1 d\xi\, \Psi_\xi^{-1} \operatorname{Tr}\Bigg(& \int_0^1 \{\exp[(\tfrac{1}{2} - s)\lambda \mathbf{X}_\xi]\} \mathbf{X}_2 \exp[-(\tfrac{1}{2} - s)\lambda \mathbf{X}_\xi]\, ds\, A_\xi^\dagger A_\xi \\
& + \Bigg[(\exp \tfrac{1}{2}\lambda \mathbf{X}_\xi)(1/i) \int_0^t U_\xi^\dagger(s) \mathscr{H}_2(s) U_\xi(s)\, ds \\
& \qquad \times \exp(-\tfrac{1}{2}\lambda \mathbf{X}_\xi) + \text{h.c.}\Bigg] A_\xi^\dagger A_\xi \\
& + \Bigg\{\exp(\tfrac{1}{2}\beta^{(i)} \mathscr{H}_\xi^{(i)}) \exp(-\tfrac{1}{2}\mu \mathbf{Y}_\xi) \int_0^1 [\exp(\tfrac{1}{2}s\mu \mathbf{Y}_\xi)](\tfrac{1}{2}\mu \mathbf{Y}_\xi) \exp(-\tfrac{1}{2}s\mu \mathbf{Y}_\xi)\, ds \\
& \qquad \times \exp(\tfrac{1}{2}\mu \mathbf{Y}_\xi) \exp(-\tfrac{1}{2}\beta^{(i)} \mathscr{H}^{(i)}) + \text{h.c.}\Bigg\} B_\xi^\dagger B_\xi \\
& + \int_0^1 \{\exp[-\beta^{(i)}(\tfrac{1}{2} - s)\mathscr{H}_\xi^{(i)}]\} \mathscr{H}_2^{(i)} \exp[\beta^{(i)}(\tfrac{1}{2} - s)\mathscr{H}_\xi^{(i)}]\, ds\, B_\xi^\dagger B_\xi \Bigg)
\end{aligned} \tag{44}$$

with $B_\xi = A_\xi^\dagger$ and

$$A_\xi = \exp(-\tfrac{1}{2}\beta^{(i)} \mathscr{H}_\xi^{(i)}) \exp(\tfrac{1}{2}\mu \mathbf{Y}_\xi^{(i)}) U^\dagger(t) \exp(\tfrac{1}{2}\lambda \mathbf{X}_\xi) \tag{45}$$

Then, the following lemma can be applied to expression (45).

Lemma 5. For any Hermitian operator $\mathscr{H}$ in a Banach algebra,

$$|\operatorname{Tr} \mathscr{H} Q^\dagger Q| \leqslant \|\mathscr{H}\| \operatorname{Tr} Q^\dagger Q \tag{46}$$

Note that all the operators in front of $A_\xi^\dagger A_\xi$ and $B_\xi^\dagger B_\xi$ in (44) are Hermitian for real λ. Such a rearrangement of operators is one of the key points of the present paper. For example, the first prefactor of $A_\xi^\dagger A_\xi$ in (44) is shown to be Hermitian as follows:

$$\begin{aligned}
\left\{\int_0^1 \exp[(\tfrac{1}{2} - s)\lambda \mathbf{X}_\xi] \mathbf{X}_2 \exp[-(\tfrac{1}{2} - s)\lambda \mathbf{X}_\xi]\, ds\right\}^\dagger &= \tilde{\mathbf{X}}_2^\dagger \\
= \int_0^1 \exp[-(\tfrac{1}{2} - s)\lambda \mathbf{X}_\xi] \mathbf{X}_2 \exp[(\tfrac{1}{2} - s)\lambda \mathbf{X}_\xi]\, ds &= \tilde{\mathbf{X}}_2
\end{aligned} \tag{47}$$

where the last equality is obtained by the transformation of variables $s' = 1 - s$.

Thus, by the help of Lemma 5, together with the property that

$$\mathrm{Tr}\, A_\xi^\dagger A_\xi = \mathrm{Tr}\, B_\xi^\dagger B_\xi = \Psi_\xi \tag{48}$$

we obtain the upper bound of Δ_n in the form

$$\begin{aligned}
|\Delta_n| \leqslant \int_0^1 d\xi \Bigg(& \left\| \int_0^1 \{\exp[(\tfrac{1}{2} - s)\lambda \mathbf{X}_\xi]\} \mathbf{X}_2 \exp[-(\tfrac{1}{2} - s)\lambda \mathbf{X}_\xi]\, ds \right\| \\
&+ \left\| [\exp(\tfrac{1}{2}\lambda \mathbf{X}_\xi)] i \left[\int_0^t U_\xi^\dagger(s) \mathscr{H}_2(s) U_\xi(s)\, ds \right] \exp(-\tfrac{1}{2}\lambda \mathbf{X}_\xi) + \text{h.c.} \right\| \\
&+ |\mu| \Bigg\| [\exp(\tfrac{1}{2}\beta^{(i)} \mathscr{H}_\xi^{(i)}) \exp(-\tfrac{1}{2}\mu \mathbf{Y}_\xi)] \\
&\times \left[\int_0^1 [\exp(\tfrac{1}{2} s\mu \mathbf{Y}_\xi)] \mathbf{Y}_2 \exp(-\tfrac{1}{2} s\mu \mathbf{Y}_\xi)\, ds \right] \\
&\times \exp(\tfrac{1}{2}\mu \mathbf{Y}_\xi) \exp(-\tfrac{1}{2}\beta^{(i)} \mathscr{H}_\xi^{(i)}) \Bigg\| \\
&+ \beta^{(i)} \left\| \int_0^1 \{\exp[-\tfrac{1}{2}\beta^{(i)}(\tfrac{1}{2} - s) \mathscr{H}_\xi^{(i)}]\} \mathscr{H}_2^{(i)} \exp[\tfrac{1}{2}\beta^{(i)}(\tfrac{1}{2} - s)\mathscr{H}_\xi^{(i)}]\, ds \right\| \Bigg)
\end{aligned} \tag{49}$$

That is, we have

$$|\Delta_n| \leqslant \Omega_2 c_n \tag{50}$$

where

$$\begin{aligned}
c_n = \max_{\mathbf{r}} \Bigg(& \int_0^1 \|\{\exp[(\tfrac{1}{2} - s)\lambda \mathbf{X}_\xi]\} \mathbf{X}(\mathbf{r}) \exp[-(\tfrac{1}{2} - s)\lambda \mathbf{X}_\xi]\|\, ds \\
&+ \left\| [\exp(\tfrac{1}{2}\lambda \mathbf{X}_\xi)] \int_0^1 i U_\xi^\dagger \mathscr{H}(\mathbf{r}, s) U_\xi(s)\, ds \exp(-\tfrac{1}{2}\lambda \mathbf{X}_\xi) + \text{h.c.} \right\| \\
&+ |\mu| \Bigg\| [\exp(\tfrac{1}{2}\beta^{(i)} \mathscr{H}_\xi^{(i)}) \exp(-\tfrac{1}{2}\mu \mathbf{Y}_\xi)] \\
&\times \int_0^1 [\exp(\tfrac{1}{2} s\mu \mathbf{Y}_\xi)] \mathbf{Y}(\mathbf{r}) \exp(-\tfrac{1}{2} s\mu \mathbf{Y}_\xi)\, ds \\
&\times \exp(\tfrac{1}{2}\mu \mathbf{Y}_\xi) \exp(-\tfrac{1}{2}\beta^{(i)} \mathscr{H}_\xi^{(i)}) \Bigg\| \\
&+ \beta^{(i)} \left\| \int_0^1 \{\exp[-\beta^{(i)}(\tfrac{1}{2} - s) \mathscr{H}_\xi^{(i)}]\} \mathscr{H}(\mathbf{r}) \exp[\tfrac{1}{2}\beta^{(i)}(\tfrac{1}{2} - s)\mathscr{H}_\xi^{(i)}]\, ds \right\| \Bigg)
\end{aligned} \tag{51}$$

In the region where $|\lambda| < \lambda_0$, $|\mu| < \mu_0, t < t_0$, and $\beta^{(i)} < \beta_0^{(i)}$ in which c_n is finite, we obtain (20), and consequently we arrive at Theorem 2. Definite values of λ_0, μ_0, t_0, and $\beta_0^{(i)}$ can be easily obtained by the method of Robinson,[16,12] but they are rather complicated and are omitted here.

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